Orbits I

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Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, where necessary.

1. The space shuttle orbits at a height of 350 km above the Earth's surface. If the Earth has a mass of 6.0×10^{24} kg and a radius of 6.4×10^6 m, calculate

   (a) the speed of the shuttle in this orbit,
   For the space shuttle, mass m in orbit (assuming a circular orbit) around the Earth of mass M, the centripetal force is the force of gravity, given by Newton's law:

$$\begin{aligned}
\frac{mv^2}{r} &= G\frac{mM}{r^2} \\
v &= \sqrt{\frac{GM}{r}} \\
&= \sqrt{\frac{6.67 \times 10^{-11}\,\text{N m}^2\ \text{kg}^{-2} \times 6.0 \times 10^{24}\,\text{kg}}{(6.4 \times 10^6 + 350 \times 10^3)\text{m}}} \\
&= 7700\sqrt{\frac{\cancel{\text{kg}}\,\cancel{\text{m}}\,\text{s}^{-2}\ \text{m}^2\ \cancel{\text{kg}^{-2}}\ \cancel{\text{kg}}}{\cancel{\text{m}}}}, \text{ since N} \equiv \text{kg m s}^{-2} \\
&= 7700\,\text{m s}^{-1}.
\end{aligned}$$

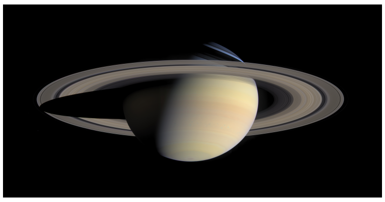
   (b) the time taken for one orbit,
   Using the familiar equation for distance, speed and time, $v = \frac{s}{t}$:

$$\begin{aligned}
v &= \frac{2\pi r}{T} \\
T &= \frac{2\pi r}{v}, \text{ and now using } v = \sqrt{\frac{GM}{r}}, \\
&= 2\pi r\sqrt{\frac{r}{GM}} \\
&= \sqrt{\frac{4\pi^2 r^3}{GM}}, \text{ which reassuringly agrees with Kepler's 3rd law that } T^2 \propto r^3! \\
&= \sqrt{\frac{4 \times \pi^2 \times (6.4 \times 10^6 + 350 \times 10^3\text{m})^3}{6.67 \times 10^{-11}\,\text{N m}^2\ \text{kg}^{-2} \times 6.0 \times 10^{24}\,\text{kg}}} \\
&= 5500\sqrt{\frac{\cancel{\text{m}^3}}{\cancel{\text{kg}}\,\cancel{\text{m}}\,\text{s}^{-2}\ \cancel{\text{m}^2}\ \cancel{\text{kg}^{-2}}\ \cancel{\text{kg}}}} \\
&= 5500\,\text{s, or 1 hours 32 min.}
\end{aligned}$$

(c) the angular velocity of this orbit.
There are 2π radians in a full circle, and this takes a time T:

$$\begin{aligned}
\omega &= \frac{2\pi}{T} \\
&= 2\pi\sqrt{\frac{GM}{4\pi^2 r^3}} \\
&= \sqrt{\frac{GM}{r^3}} \\
&= \sqrt{\frac{6.67\times 10^{-11}\,\mathrm{N\ m^2\ kg^{-2}} \times 6.0\times 10^{24}\,\mathrm{kg}}{(6.4\times 10^6 + 350\times 10^3\mathrm{m})^3}} \\
&= 1.1\times 10^{-3}(\mathrm{rad})\mathrm{s}^{-1}
\end{aligned}$$

2. The rings of Saturn consist of a vast number of small particles, each in a circular orbit. They are shown in the image below [Public Domain, from Cassini Spacecraft, NASA].



The inner edge of the inner ring is 70 000 km from the centre of the planet, and the outermost edge of the outer ring is 140 000 km from the centre. The speed of the outermost particles is $17\,\mathrm{km\ s^{-1}}$.

(a) Show that the speed, v, of a particle in orbit of radius r around a planet of mass M is given by

$$v = \sqrt{\frac{GM}{r}}.$$

For a particle of mass m orbiting Saturn (of much larger mass M) in one of the rings at a distance from the centre of the planet of R the pull of gravity is given by Newton's law $G\frac{Mm}{R^2}$. Assuming a circular orbit, this provides the centripetal force inwards of $\frac{mv^2}{R}$ which keeps it going around in a circle:

$$\begin{aligned}
\frac{mv^2}{r} &= G\frac{mM}{r^2} \\
v &= \sqrt{\frac{GM}{r}}
\end{aligned}$$

(b) Determine the mass of Saturn.
Rearranging the equation given:

$$M = \frac{rv^2}{G}$$

G is known (the gravitational constant, $6.67\times 10^{-11}\,\mathrm{N\ m^2\ kg^{-2}}$), and the only particles for which r and v are given in the question are the outermost particles, for which (remembering to put everything into m to agree with the units that we're using for G):

$$\begin{aligned}
M &= \frac{140\,000\times 10^3\,\mathrm{m}\times(17\times 10^3\,\mathrm{m\ s^{-1}})^2}{6.67\times 10^{-11}\,\mathrm{N\ m^2\ kg^{-2}}} \\
&= 6.1\times 10^{26}\frac{\cancel{\mathrm{m}}\cancel{\mathrm{m}^2}\cancel{\mathrm{s}^{-2}}}{\mathrm{kg}\,\cancel{\mathrm{m}\mathrm{s}^{-2}}\,\cancel{\mathrm{m}^2}\,\mathrm{kg}^{-2}} \\
&= 6.1\times 10^{26}\frac{1}{\mathrm{kg}^{-1}} \\
&= 6.1\times 10^{26}\,\mathrm{kg}.
\end{aligned}$$

Is this right? Well, we were given the mass of the Earth as 6×10^{24} kg in the first question, so we can compare these and see that our answer puts the mass of Saturn at 100 times that of the Earth. Since Saturn is a gas giant, this seems about right.

(c) How long does it take for the outermost particles to complete an orbit?

$$\begin{aligned} T &= \frac{2\pi r}{v} \\ &= \frac{2 \times \pi \times 140\,000\,\text{km}}{17\,\text{km}\,\text{s}^{-1}} \quad \text{NB we can stay in km throughout here} \\ &= 52\,000\,\text{s, or } 14.4 \text{ Earth hours.} \end{aligned}$$

(d) Calculate the orbital speed of the particles nearest to Saturn.
No need to use all the constants and so forth here (though you can if you want, it may lead to rounding errors from the mass of Saturn result earlier). Much better to use scaling:

$$\begin{aligned} v &= \sqrt{\frac{GM}{r}} \\ v &\propto r^{-\frac{1}{2}} \\ \frac{v_{\text{inner}}}{v_{\text{outer}}} &= \left(\frac{r_{\text{inner}}}{r_{\text{outer}}}\right)^{-\frac{1}{2}} \\ v_{\text{inner}} &= v_{\text{outer}} \left(\frac{r_{\text{inner}}}{r_{\text{outer}}}\right)^{-\frac{1}{2}} \\ &= 17\,\text{km}\,\text{s}^{-1} \left(\frac{70\,000\,\cancel{\text{km}}}{140\,000\,\cancel{\text{km}}}\right)^{-\frac{1}{2}} \\ &= 24\,\text{km}\,\text{s}^{-1}. \end{aligned}$$

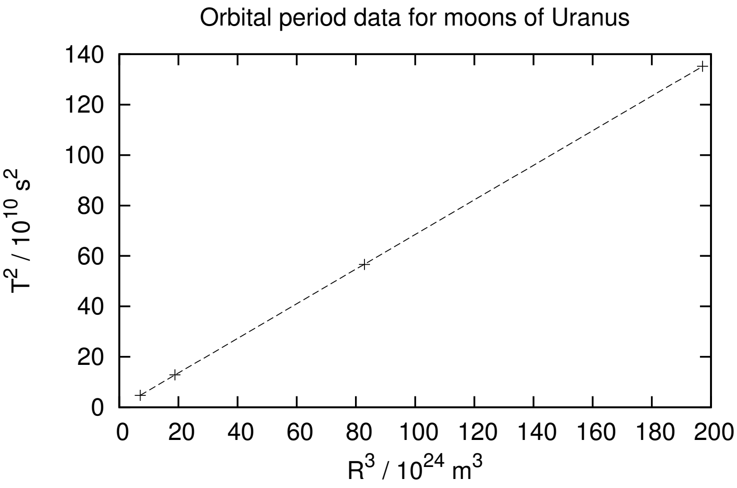
3. (a) Use the following data about four of the moons of Uranus to plot a suitable graph to test Kepler's Law:

Orbit time / hour	60.5	99.5	209	323
Orbit radius / 10^3 km	192	266	436	582

Kepler's third law states that the square of the orbital period is directly proportional to the cube of the semi-axis major of the ellipse which an orbiting body follows as its path of motion, i.e.

$$T^2 \propto R^3,$$

meaning that if T^2 is plotted against R^3, a straight line will result. I have chosen to make the units the SI s and m to make calculations more straightforward, but provided that a consistent approach is used, hours and km (or any other choice) will give the same answers.



(b) If a further moon of Uranus were discovered with a period of 170 hours, what would be its orbital radius.
This would be a T^2 value of $37.5 \times 10^{10}\,\mathrm{s}^2$, which from the line of best fit on the graph has a R^3 value of $54.8 \times 10^{24}\,\mathrm{m}^3$. This gives its orbital radius as $380 \times 10^3\,\mathrm{km}$, in the units used in the table, which is between the moons with just longer and shorter orbital periods.

(c) Use the graph to estimate a value for the mass of Uranus.
The line of best fit is a straight line fitted to the data, and therefore has a for given by

$$T^2 = aR^3 + b,$$

where a and b are constants. Calculating (or measuring) these constants, we have

$$T^2 = (6.86 \pm 0.01) \times 10^{-15}\,\mathrm{s}^2\,\mathrm{m}^{-3} R^3 - (1.4 \pm 0.8) \times 10^9\,\mathrm{s}^2.$$

Comparing this with the for for Kepler's 3rd law,

$$T^2 = \left(\frac{4\pi^2}{GM}\right) R^3,$$

we can identify the constant a:

$$\begin{aligned}
a &= \frac{4\pi^2}{GM} \\
M &= \frac{4\pi^2}{Ga} \\
&= \frac{4 \times \pi^2}{6.67 \times 10^{-11}\,\mathrm{N\ m^2\ kg^{-2}} \times 6.86 \times 10^{-15}\,\mathrm{s^2\ m^{-3}}} \\
&= 8.62 \times 10^{25}\,\frac{1}{\mathrm{kg}\,\cancel{\mathrm{m\,s^{-2}}}\,\cancel{\mathrm{m^2}}\,\mathrm{kg^{-2}}\,\cancel{\mathrm{s^2}}\,\cancel{\mathrm{m^{-3}}}} \\
&= 8.62 \times 10^{25}\,\mathrm{kg}
\end{aligned}$$

Is this reasonable? It is 15 times the mass of the Earth, but much smaller than that of Saturn, and so seems consistent with previous results in this homework.