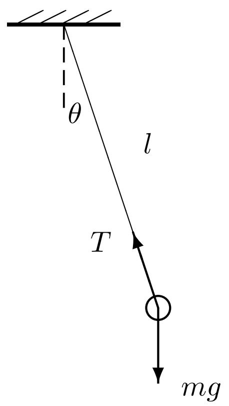
The Simple Pendulum

A.C. Norman

September, 2010

The system known to physics as the simple pendulum is an imaginary, idealized object. It comprises a bob which swings without losing energy in a vertical two-dimensional plane on a massless, inextensible string which always remains taut.

As we shall see, the mathematics even of this seemingly innocuous problem is rich and complex; in general, the equations governing the motion are non-linear and cannot be solved analytically. However, for small angles of swing, the pendulum will approximate the behaviour of another fictional system—the Harmonic Oscillator—which we can determine exactly.



To obtain the equation of motion, we use an angular version of Newton's second law:

$$T = I\ddot{\theta}, \tag{1}$$

where T is the torque, I the moment of inertia, and $\ddot{\theta}$ the angular acceleration.
In this case, the moment of inertia is $I = ml^2$, and The return torque produced by gravity is

$$T = -(mg\sin\theta)l$$

when the angular displacement is θ.

We can see that in this case (1) becomes

$$\begin{aligned} ml^2\ddot{\theta} &= -mgl\sin\theta \\ \ddot{\theta} &= -\frac{g}{l}\sin\theta \end{aligned} \tag{2}$$

1 Small angles: Simple Harmonic Motion

For small angles, $\theta \ll 1$, we make the approximation $\sin\theta \approx \theta$. This is based on the standard series expansion for $\sin\theta$, which is

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots,$$

and allows us to see that, to first order, the error which this introduces is proportional to θ^3.

Substituting this approximation into the equation of motion (2) yields a second-order ordinary differential equation which we already know how to solve: it has the form of the harmonic oscillator (3).

$$\begin{aligned} \ddot{\theta} + \frac{g}{l}\theta &= 0 \\ \ddot{\psi} + \omega_0^2\psi &= 0 \end{aligned} \tag{3}$$

It is easy to verify, by differentiation and substitution, that (3) is satisfied by an expression of the form

$$\psi(t) = A\cos(\omega_0 t + \phi) \tag{4}$$

where A is any constant length (the amplitude) and ϕ is any constant angle (the initial phase). A quantity which varies as ψ does with time is said to vary harmonically, and a vibration which it describes is termed harmonic motion. ω_0 is the angular frequency of such a vibration.

The controlling quantity in (4) is the *phase angle* $\omega_0 t + \phi$, sometimes simply called the phase. The phase angle increases uniformly with time, but situations with their phase angles differing by any multiple of 2π are physically indistinguishable. This means that the harmonic motion is periodic: a sequence of repeated cycles. The interval of repetition is known as the period T given by $\omega_0 T = 2\pi$, and the reciprocal of this is the number of cycles per unit time, which we call the frequency $\nu_0 = 1/T = \omega_0/2\pi$.

In our case, identifying the equation of motion we have written with the form for the harmonic oscillator gives

$$\omega_0 = \sqrt{\frac{g}{l}},$$

and so we can see that the time period T for one oscillation of the pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{5}$$

1.1 Pendulum rule of thumb

Rearranging (5) to have l as the subject of the formula, we find that

$$l = \frac{g}{\pi^2}\frac{T^2}{4}.$$

So long as we measure in metres and seconds, we can say that $g \approx \pi^2$, so

$$l \approx \frac{T^2}{4},$$

"the length of a pendulum (in metres) is approximately one quarter of the square of the time period (in seconds)," which is a handy rule-of-thumb for everyday use.

2 Arbitrary angles: Elliptic Integrals

As we have already seen, it is not possible to turn the equation of motion for the simple pendulum into that of a harmonic oscillator exactly. This difficulty arises because gravity always acts vertically, whereas the motion of the pendulum is rotational; the return torque is not proportional to θ, but has a more complicated dependence, meaning the system is non-linear.

We shall now press ahead and see how far we can progress with the true equation of motion (2) without approximating.

Firstly, we multiply both sides of the original equation by $\dot{\theta}$.

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$
$$\dot{\theta}\ddot{\theta} = -\frac{g}{l}\dot{\theta}\sin\theta$$

We can then use the chain rule in reverse to identify that $\frac{d}{dt}(\dot{\theta}^2) = 2\dot{\theta}\ddot{\theta}$ and $\frac{d}{dt}\cos\theta = -\dot{\theta}\sin\theta$, yielding

$$\frac{1}{2}\frac{d}{dt}(\dot{\theta}^2) = \omega_0^2\frac{d}{dt}\cos\theta.$$

Now integrating both sides with respect to t, we see that

$$\dot{\theta}^2 = 2\omega_0^2\cos\theta + c, \tag{6}$$

where c is a constant of integration. This constant can be found by noting that at the maximum extent of the oscillation, which we shall henceforth denote by θ_0, the bob comes to an instantaneous halt, and so $\dot{\theta} = 0$. Putting these values into (6) gives $c = -2\omega_0\cos\theta_0$, and so (6) becomes

$$\dot{\theta} = \omega_0\sqrt{2}(\cos\theta - \cos\theta_0)^{\frac{1}{2}}. \tag{7}$$

This is a separable first-order ordinary differential equation, and so we ought now to proceed by rearranging the equation so that the terms in θ and t appear on opposite sides of the equation (i.e. are separated) and integrating:

$$\int \frac{d\theta}{\omega_0\sqrt{2}(\cos\theta - \cos\theta_0)^{\frac{1}{2}}} = \int dt.$$

Finding the solution $\theta(t)$ that satisfies (7) depends only on the ease with which the integrals on each side of the equation above can be evaluated – one is trivial, the other cannot be done analytically.

Let us press on by choosing to integrate definitely over a quarter period (chosen since $\theta < \theta_0 \Rightarrow \dot{\theta} > 0$).

$$\int_0^{\frac{T}{4}} dt = \int_0^{\theta_0} \frac{d\theta}{\omega_0\sqrt{2}(\cos\theta - \cos\theta_0)^{\frac{1}{2}}}$$
$$\frac{T}{4} = \frac{1}{\omega_0\sqrt{2}}\int_0^{\theta_0} \frac{d\theta}{(\cos\theta - \cos\theta_0)^{\frac{1}{2}}} \tag{8}$$

We shall now proceed to turn the above expression into an equivalent form which has been studied in some detail. Firstly, we make use of the double angle formula

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 1 - 2\sin^2 A$$

to rewrite the integral in (8) as

$$\int_0^{\theta_0} \frac{d\theta}{\left(1 - 2\sin^2\frac{\theta}{2} - 1 + 2\sin^2\frac{\theta_0}{2}\right)^{\frac{1}{2}}}$$
$$= \int_0^{\theta_0} \frac{d\theta}{\sqrt{2}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}.$$

We now make a change of variable $\theta \to \phi$, where ϕ is defined by

$$\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\sin\phi.$$

Differentiating both sides of this with respect to θ,

$$\frac{d}{d\theta}\sin\frac{\theta}{2} = \frac{d}{d\theta}\sin\frac{\theta_0}{2}\sin\phi$$

$$\begin{aligned}\frac{1}{2}\cos\frac{\theta}{2} &= \frac{d}{d\phi}\frac{d\phi}{d\theta}\sin\frac{\theta_0}{2}\sin\phi \\ &= \frac{d\phi}{d\theta}\sin\frac{\theta_0}{2}\cos\phi\end{aligned}$$

$$d\theta\cos\frac{\theta}{2} = 2d\phi\sin\frac{\theta_0}{2}\cos\phi,$$

and we also note that where $\theta = \theta_0$, $\sin\phi = 1 \Rightarrow \phi = \frac{\pi}{2}$.

Using these facts allows us to make the substitution, rewriting the integral as

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \frac{2\sin\frac{\theta_0}{2}\cos\phi\, d\phi}{\sqrt{2}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta_0}{2}\sin^2\phi\right)^{\frac{1}{2}}\cos\frac{\theta}{2}} \\ =&\int_0^{\frac{\pi}{2}} \frac{\sqrt{2}\sin\frac{\theta_0}{2}\cos\phi\, d\phi}{\sin\frac{\theta_0}{2}\left(1-\sin^2\phi\right)^{\frac{1}{2}}\left(\cos^2\frac{\theta}{2}\right)^{\frac{1}{2}}} \\ =&\int_0^{\frac{\pi}{2}} \frac{\sqrt{2}\cos\phi\, d\phi}{\cos\phi\left(1-\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}} \\ =&\sqrt{2}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\left(1-\sin^2\frac{\theta_0}{2}\sin^2\phi\right)^{\frac{1}{2}}}.\end{aligned} \tag{9}$$

The integral in (9) is known as a complete elliptic integral of the first kind, denoted in mathematics by K, which has been studied in some detail by mathematicians. The integral is defined in many texts as

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\left(1-m\sin^2\theta\right)^{\frac{1}{2}}} = \int_0^1 \frac{dt}{\left[\left(1-t^2\right)\left(1-mt^2\right)\right]^{\frac{1}{2}}},$$

and it can be expressed as a power series

$$K(m) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}n!^2}\right]^2 m^{2n} = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 m^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 m^6 + \cdots\right],$$

which is valid for $|m| < 1$. In our case, $m = \sin^2\frac{\theta_0}{2}$. Our conclusion, therefore, is that

$$\begin{aligned}\frac{T}{4} &= \frac{1}{\omega_0\sqrt{2}}\sqrt{2}\,K(m) \\ T &= T_0\frac{K(m)}{\pi/2}.\end{aligned}$$